Uzavřené a exaktní formy

Def:

ω je uzavřená $\equiv$ $d\omega = 0$ $\qquad \Omega_c^p M \quad \Omega_c M$

ω je exaktní $\equiv \exists \sigma \ \omega = d\sigma$ $\qquad \Omega_e^p M \quad \Omega_e M$

neboli

$\Omega_c^p M = \ker d \qquad \Omega_e M = \operatorname{img} d$

platí

ω exaktní $\Rightarrow$ ω uzavřená

jak je to naopak? - ne vždy → ...

→ deRhamova kohomologie

platí

$\Omega_c M$ je podalgebra ΩM

$d\omega = 0 \quad d\sigma = 0 \quad \Rightarrow \quad d(\omega \wedge \sigma) = d\omega \wedge \sigma \pm \omega \wedge d\sigma = 0$

$\Omega_e M$ je ideál v $\Omega_c M$

$\omega = dk \quad d\sigma = 0 \quad \Rightarrow \quad \omega \wedge \sigma = dk \wedge \sigma = d(k \wedge \sigma)$

deRhamova kohomologie

$H^p_{dR}(M) = \Omega_c^p M / \Omega_e^p M$

abelovská grupa s oper. sčítání

$\wedge$ operace mezi kohomolog. gr. → kohomolog. algebra $H_{dR}(M)$

Bettiho čísla

$\dim H^p_{dR}(M) = b^p(M) < \infty$

Eulerova charakteristika

$$\chi(M) = \sum_{p=0}^{d} (-1)^p \, b^p(M)$$

Počet komponent

$b^0(M) =$ počet komponent M

$\Omega_c^0 M$ - lok. konst. fce $\quad \Omega_e^0 M$ triviální $\Rightarrow$

$H^0_{dR}(M) = \operatorname{Lin}(1_K,\ K$ komponenty $M) \qquad \dim H^0_{dR}(M) =$ počet komponent

Př: Kohomologické gr. $H^0(M)$

M souvislá kompaktní varieta s metr. g

$\Omega^0 M = \mathcal{F} M$

$\Delta f = 0 \iff df = 0 \quad \underset{\text{triviální}}{\delta f = 0} \iff f = \text{konst}$

$\Downarrow$

$\mathcal{H}^0(M) = \operatorname{Lin}(1) \qquad H^0(M) = \mathbb{R}$

platí

$\Delta f = \delta d f$

Hodgeho rozštěpení

$f = \qquad \frac{\Omega}{V}$

kde

$\Omega = \int_M * f$

M kompaktní varieta s komponentami M_k $k = 1 \dots K$

$\mathcal{H}^0(M) = \operatorname{Lin}(1_1, 1_2, \dots 1_K) \quad H^0(M) = \mathbb{R}^K$

$f = \qquad \sum_k \frac{\Omega_k}{V_k}$

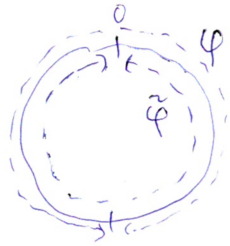
$\Omega_k = \int_{M_k} * f \qquad V_k = \int_{M_k} \varepsilon$

příklady:

Př: $\mathbb{R}^n$

$d\omega = 0 \iff \exists \varphi \quad \omega = d\varphi$

Př: S^1



$\varphi \in (-\pi, \pi)$

$\tilde{\varphi} \in (0, 2\pi)$

$\tilde{\varphi} = \varphi \qquad \varphi, \tilde{\varphi} \in (0, \pi)$

$\tilde{\varphi} = \varphi + 2\pi \qquad \varphi \in (-\pi, 0) \quad \tilde{\varphi} \in (\pi, 2\pi)$

$\Phi = d\varphi = d\tilde{\varphi}$ kde lze definovat

Φ globální hladká 1-forma

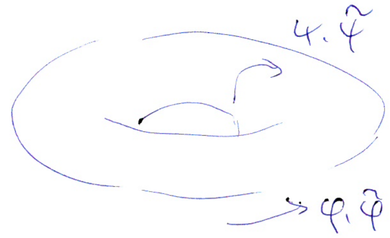
$\varphi, \tilde{\varphi}$ globálně nehladké

Φ uzavřená, ale neexaktní

$H^1_{dR}(S^1) = \mathbb{R} \qquad [\Phi] = \Phi + \Omega^1_e S^1$

$b^0(S^1) = 1 \qquad b^1(S^1) = 1 \qquad \chi(S^1) = 0$

Př: $T^2 = S^1 \times S^1$



dvě cyklické souř.

$\varphi, \tilde{\varphi} \longrightarrow$ 1-forma $\Phi = d\varphi = d\tilde{\varphi}$

$\psi, \tilde{\psi} \longrightarrow$ 1-forma $\Psi = d\psi = d\tilde{\psi}$

$H^1_{dR}(T^2)$ generováno $[\Phi], [\Psi]$

$H^2_{dR}(T^2)$ generováno $[\Phi \wedge \Psi]$

$b^0(T^2) = 1 \qquad b^1(T^2) = 2 \qquad b^2(T^2) = 1 \qquad \chi(T^2) = 0$

Př: S^n

platí

$H^0_{dR}(S^n) = \mathbb{R} \qquad H^n_{dR}(S^n) = \mathbb{R}$ ostatní triviální

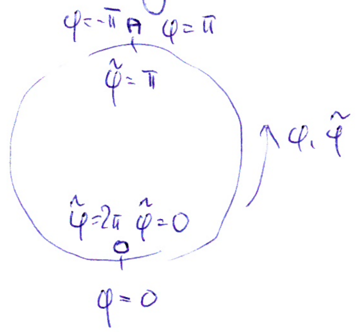
$$b^p(M) = \begin{cases} 1 & p = 0, n \\ 0 & p \neq 0, n \end{cases} \qquad \chi(S^n) = \begin{cases} 2 & n \text{ sudé} \\ 0 & n \text{ liché} \end{cases}$$

Př: kohomologické gr. variety S^1

souřadnice

$\varphi \in (-\pi, \pi)$

$\tilde\varphi \in (0, 2\pi)$



$\Phi = d\varphi$ na $S^1 \setminus \{A\}$

$= d\tilde\varphi$ na $S^1 \setminus \{O\}$

$\Phi \in \mathcal{A}^1 S^1$

metrika

$g = \Phi\Phi = d\varphi^2 = d\tilde\varphi^2$

$\varepsilon = \Phi \qquad *\Phi = 1$

$d\Phi = 0 \qquad \delta\Phi = -*d*\Phi = 0 \qquad \Rightarrow \Delta\Phi = 0$

harmoniky

$\omega \in \mathcal{A}^1 S^1 \Rightarrow \omega = \omega_\varphi \Phi$ $\quad \omega_\varphi$ period. ve φ

$\Delta\omega = (\Delta\omega_\varphi)\Phi \quad \Leftarrow$ např. $\Delta = \nabla^2 \quad \nabla\Phi = 0$

$\Delta\omega = 0 \Leftrightarrow \Delta\omega_\varphi = 0 \Leftrightarrow \omega_\varphi =$ konst

$\omega \in \mathcal{H}^1(S^1) \Leftrightarrow \omega = \Omega\Phi$

$\mathcal{H}^1(S^1) = \mathrm{Lin}(\Phi) \qquad H^1(S^1) = \mathbb{R}$

Hodge

$\omega \in \mathcal{A}^1 S^1 \qquad d\omega = 0$

$\omega = d\alpha + \frac{\Omega}{2\pi}\Phi \qquad \alpha \in \mathcal{F} S^1 \qquad \frac{\Omega}{2\pi} \in \mathcal{H}^1(S^1)$

kde

$\Omega = \int_{S^1}\omega \quad \Leftarrow \quad \int_{S^1}\omega = \underbrace{\int_{S^1} d\alpha}_{0} + \frac{\Omega}{2\pi}\underbrace{\int_{S^1}\Phi}_{2\pi} = \Omega$

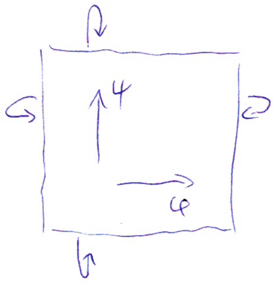
α řešení rov. $\Delta\alpha = \delta\omega$

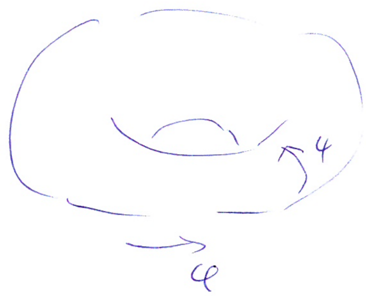
Př: kohomologické gr. variety T^2

souřadnice

φ, ψ nespojité

viz S^1





$\Phi = d\varphi$ lze globálně

$\Psi = d\psi$ na celé T^2

metrika

$$g = \Phi\Phi + \Psi\Psi = d\varphi^2 + d\psi^2$$

$$\varepsilon = \Phi \wedge \Psi \qquad *\Phi = \Psi \qquad *\Psi = -\Phi \qquad d\Phi = d\Psi = 0$$

$$\Delta = \nabla^2 \Leftarrow R = 0 \qquad \nabla\Phi = \nabla\Psi = 0 \qquad \delta\Phi = \delta\Psi = 0$$

$$\delta = *d*$$

harmonicky $\mathcal{H}^1(T^2)$

$\omega \in \mathcal{A}^1 T^2 \qquad \omega = \omega_\varphi \Phi + \omega_\psi \Psi \qquad \omega_\varphi, \omega_\psi$ periodo φ, ψ

$$\Delta\omega = (\omega_{\varphi,\varphi\varphi} + \omega_{\varphi,\psi\psi})\Phi + (\omega_{\psi,\varphi\varphi} + \omega_{\psi,\psi\psi})\Psi$$
$$= (\Delta\omega_\varphi)\Phi + (\Delta\omega_\psi)\Psi$$

proof:

$\Delta = \nabla^2 \qquad \nabla\Phi = \nabla\Psi = 0$ nebo

$$d\omega = \omega_{\varphi,\psi}\Psi\wedge\Phi + \omega_{\psi,\varphi}\Phi\wedge\Psi = (\omega_{\psi,\varphi} - \omega_{\varphi,\psi})\varepsilon$$

$$*d\omega = \omega_{\psi,\varphi} - \omega_{\varphi,\psi}$$

$$d*d\omega = (\omega_{\psi,\varphi\varphi} - \omega_{\varphi,\psi\varphi})\Phi + (\omega_{\psi,\varphi\psi} - \omega_{\varphi,\psi\psi})\Psi$$

$$\delta d\omega = *d*d = (\omega_{\varphi,\psi\psi} - \omega_{\psi,\varphi\psi})\Phi + (\omega_{\psi,\varphi\varphi} - \omega_{\varphi,\psi\varphi})\Psi$$

$$*\omega = -\omega_\psi\Phi + \omega_\varphi\Psi$$

$$d*d*\omega = (\omega_{\varphi,\varphi\varphi} + \omega_{\psi,\psi\varphi})\Phi + (\omega_{\psi,\psi\psi} + \omega_{\varphi,\varphi\psi})\Psi$$

$$\Delta\omega = (\omega_{\varphi,\varphi\varphi} + \omega_{\varphi,\psi\psi})\Phi + (\omega_{\psi,\varphi\varphi} + \omega_{\psi,\psi\psi})\Psi$$

$\omega \in \mathcal{H}^1(T^2) \Leftrightarrow \omega_\varphi = $ konst $\quad \omega_\psi = $ konst

$\mathcal{H}^1(T^2) = Lin(\Phi, \Psi) \qquad H^1(T^2) = \mathbb{R}^2$

Hodge

$\omega \in A^1 M \qquad \exists\, \alpha \in \mathcal{F}M \qquad \beta \in A^2 M \qquad \Omega, \Delta \in \mathbb{R}$

$$\omega = d\alpha + \delta\beta + \frac{\Omega}{2\pi}\Phi + \frac{\Delta}{2\pi}\Psi$$

$$= df + * dh + \frac{\Omega}{2\pi}\Phi + \frac{\Delta}{2\pi}\Psi$$

$f, h \in \mathcal{F}M \qquad h = *\beta \qquad \Omega, \Delta \in \mathbb{R}$

pro uzavřené ($\delta\omega = 0$) máme

$$\Omega = \int_{c^1} \omega \qquad [c_1], [c_2] \text{ tvoří duální bázi k } \frac{1}{2\pi}\Phi, \frac{1}{2\pi}\Psi$$

$$\Delta = \int_{c^2} \omega$$

obecně:

f řeší $\Delta f = \delta\omega$

h řeší $\Delta h = * d\omega$

harmonity $\mathcal{H}^2(T^2)$

$\omega \in A^2 T^2 \qquad \omega = \omega_{\varphi\psi}\, \phi \wedge \psi$

$$\Delta\omega = (\Delta\omega_{\varphi\psi})\, \phi \wedge \psi$$

proof

$\Delta = \nabla^2 \qquad \nabla\phi = \nabla\psi = 0 \qquad$ nebo

$d\omega = 0$

$*\omega = \omega_{\varphi\psi} \qquad d*\omega = \omega_{\varphi\psi,\varphi}\,\phi + \omega_{\varphi\psi,\psi}\,\psi$

$\delta\omega = *d*\omega = -\omega_{\varphi\psi,\psi}\,\phi + \omega_{\varphi\psi,\varphi}\,\psi$

$\Delta\omega = d\delta\omega = (\omega_{\varphi\psi,\varphi\varphi} + \omega_{\varphi\psi,\psi\psi})\,\phi \wedge \psi$

$\omega \in \mathcal{H}^2(T^2) \iff \omega_{\varphi\psi} = \text{konst}$

$\mathcal{H}^2(T^2) = \mathrm{Lin}_1(\varepsilon) \qquad H^2(T^2) = \mathbb{R}$

Hodge

$\omega \in A^2 M \qquad \exists\, \alpha \in A^1 M \qquad \Omega \in \mathbb{R}$

$$\omega = d\alpha + \frac{\Omega}{(2\pi)^2}\varepsilon$$

kde

$\Omega = \int_{T^2} \omega \qquad \alpha$ řeší $\Delta\alpha = \delta\omega$